The Minimum Linear Arrangement Problem

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Outline

- Definition of the problem
- Existing accurate algorithms
- Heuristics
- Experimental evaluation of heuristics

Definition

Given a graph G(V,E) where E and V are the sets of edges and vertices respectively, a permutation π is defined as

π: V -->{1,2,...,n}.

The Minimum Linear Arrangement problem requires to find the permutation π s.t. the below objective function is minimized:

$$\sum_{(i,j)\in E} |\pi(i) - \pi(j)|$$

Applications: VLSI design, biology, graph drawing…

It has been proven that the problem in its general form is NP-complete.

So are we done?

No!

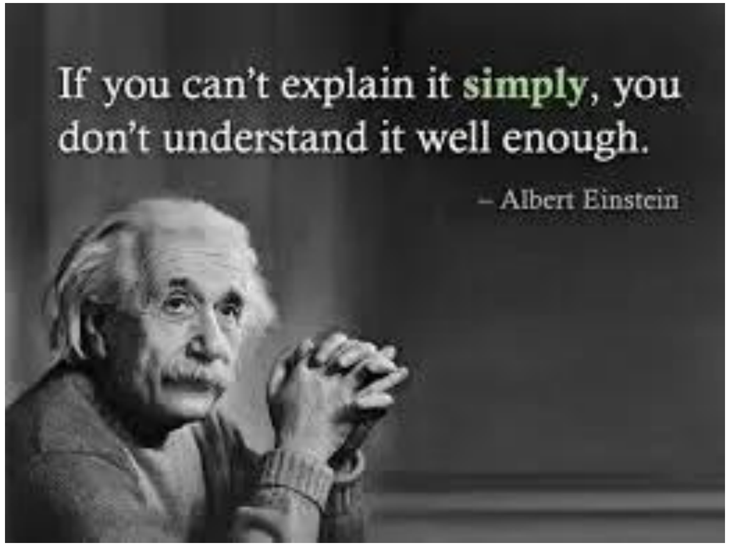
In some cases (small graphs, complete graphs, complete bipartite graphs, rooted or undirected trees), there exist algorithms running in polynomial time (maybe with some constraints in the output too).

Categorization of graphs input

- Small graphs
- Undirected trees
- Rooted trees

Summary of existing accurate algorithms

Algorithm Author	Characteristics		
	Input	*Output*	*Time complexity*
Amaral	Small number of vertices	-	exponential
Shiloach	undirected trees	-	$O(n^{2.2})$
Chung	undirected trees, rooted trees	-	$O(n^2)$ or $O(n^\lambda)$ with $\lambda > \frac{log3}{log2}$
Iordanskii	undirected trees	planar graph	$O(n)$
Alemany-Puig, Esteban, Ferrer-i-Canch	undirected trees, rooted trees	planar, projective graph	$O(n)$



Approximation techniques (heuristics)

- Spectral sequencing
- Random layout
- Normal layout
- Successive Augmentation
- Local Search
- Hillclimbing
- Spreading metrics
- …

Spectral sequencing

Proposed by Juvan and applied as follows:

- Compute matrices A(G), D(G) and subsequently the Laplacian matrix L(G).
- Find the Laplacian eigenvalues and sort them in increasing order, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.
- Compute the eigenvector $x^{(2)}$ (Fiedler vector) which corresponds to the second smallest laplacian eigenvalue λ_2.
- Determine labeling ψ^e s.t. :

  If $x^{(2)}_u \leq x^{(2)}_v$ then $\psi^e(u) \leq \psi^e(v)$

Lower bound achieved: $LB_{SO} := \lceil \lambda_2(G)\frac{n^2-1}{6} \rceil$

Local Search

- Generalized method for solving many combinatorial problems.
- Trying to find out local minimal (or maximal) by performing local changes.
- Need to specify the set of feasible solutions, the objective function and the concept of “neighborhood”.
- Simple, comprehensive and most times effective family of algorithms.

Implementation of Local Search in MLA

- Set of feasible solutions: the set of all permutations of size n (n!).
- Objective function: we saw that before,

$$\sum_{(i,j)\in E} |\pi(i) - \pi(j)|$$

- Neighborhood ??

Definition of the term “neighborhood”

- Definition 1: two layouts are neighbors if one can move from one to another by flipping the labels of any pair of nodes in the graph (version 2).
- Definition 2: Two layouts are neighbors if one can move from one to another by flipping the labels of two adjacent nodes in the graph (version 2b).
- Definition 3: Two layouts are neighbors if one can move from one to another by rotating the labels of any triplet of nodes in the graph (version 3).

But even if we specify the term of neighborhood, which will be the appropriate stopping criterion?

Experimental evaluation of heuristics

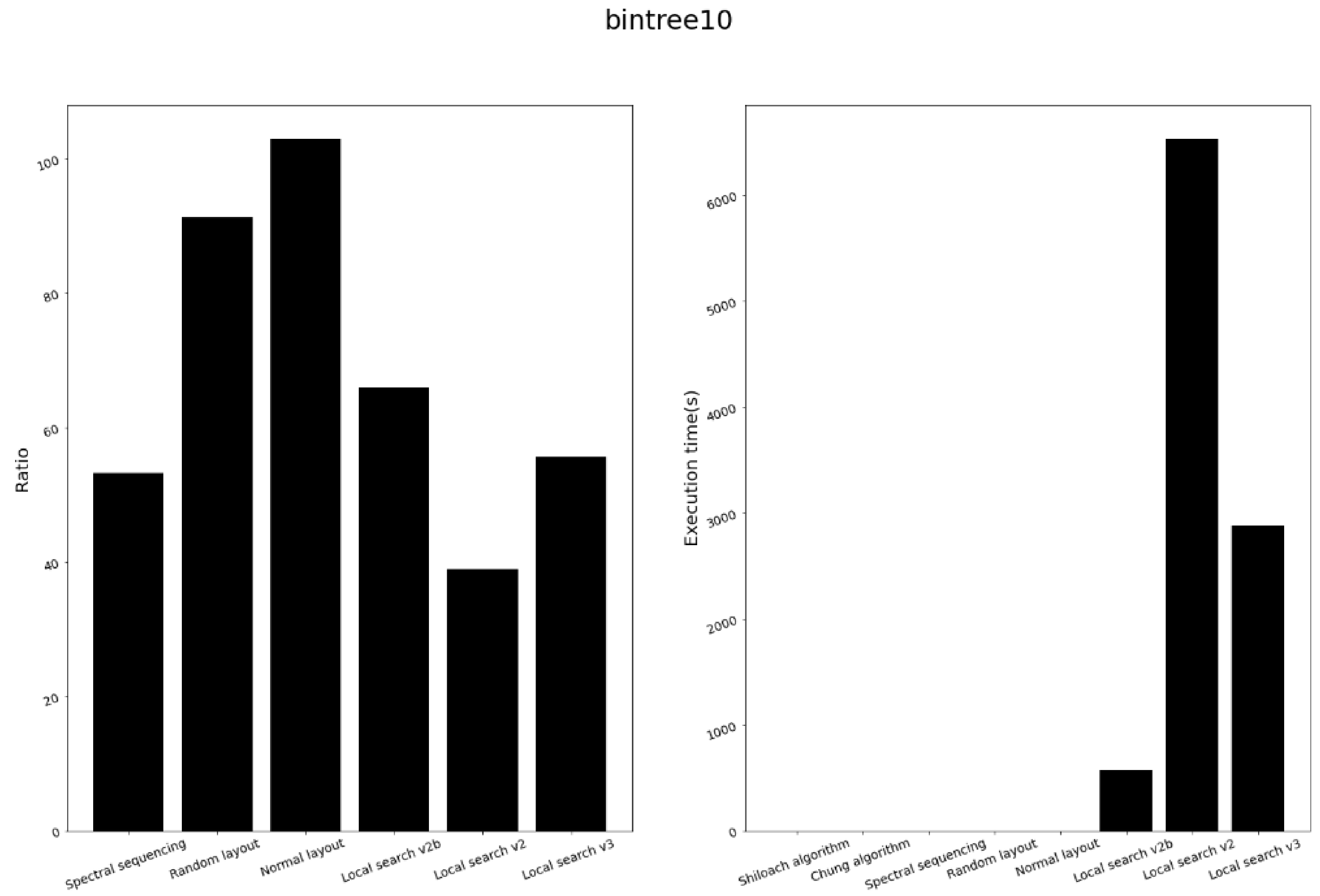
- Implemented approximation techniques: Normal layout, Random layout, Spectral Sequencing, Local Search (3 definitions of neighborhood).
- Python (Anaconda and Spyder)
- Use of Linear Arrangement Library (LAL) (Llu ́ıs Alemany-Puig, Juan Luis Esteban and Ramon Ferrer-i-Cancho) which uses the Algorithms of Chung and Shiloach.
- Comparison of results according to the quality of solution and execution time …

Input graphs of experiments

Name	Characteristics		
	Nodes	*Edges*	*Description*
bintree10	1023	1022	a complete binary tree with 10 levels
c1y	828	1749	graph from VLSI design
c2y	980	2102	graph from VLSI design
football	115	613	American Football Games
gd95c	62	144	graph from graph-drawing competitions
gd96b	111	193	graph from graph-drawing competitions
gd96c	65	125	graph from graph-drawing competitions
gd956d	180	288	graph from graph-drawing competitions
hc10	1024	5120	10-hypercube
randomA1	500	2454	Random Graph (Gilbert) with n=500 and p=0.02
randomA2	500	5004	Random Graph (Gilbert) with n=500 and p=0.04
randomA3	500	800	Random Graph (Erdos-Renyi) with n=500 and M=800
randomG1	500	2019	Random Geometric Graph with n=500 and radius=0.075
randomG2	500	5712	Random Geometric Graph with n=500 and radius=0.125

Use of properly processed real-life graphs provided by Jordi Petit and randomly generated graphs.

Evaluation of Heuristics: an example



Conclusion

- Expected bad performance of Normal and Random Layout.
- Very satisfying time and quality performance of Spectral Sequencing (but inappropriate heuristic for large graphs because of computational power - huge use of memory).
- The great interest: Local Search parameters…

Conclusion (cont.)

- Local Search: what to choose?
- It seems that the definition 1 (version 2) gives the best results referring to the approximation ratio among the 3 neighborhoods…

  but with max_iterations close to 3000-4000!

- With smaller stopping criterion, Local Search behaves worse even than Spectral Sequencing!
- With max_iterations in the range 5000-10000, Local Search v2 approaches the optimal solution…

  but it runs for hours even for graphs with 500-1000 nodes!

So what to choose?

- Tradeoff between approximation ratio and execution time.

How close are we to optimality?

- Local Search is a randomized family of algorithms.
- It depends on the selection of the initial permutation.
- Could we choose a “close enough” initial permutation so as to save time…?

and our computer too!

Questions?

Thank you for your attention!

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